数理解析研究所講究録 1236 巻 2001 年 131-135

Spacelike stationary surfaces in semi-Riemannian space forms

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Let $N_p^n(c)$ denote the n-dimensional simply connected semi-Riemannian space form of constant curvature c and index p, where we write $N^n(c)$ if $p=0$. We say that a spacelike surface in $N_p^n(c)$ is stationary if its mean curvature vector vanishes identically. We are interested in comparing the geometries of spacelike stationary surfaces in $N_p^n(c)$ of various index p.

We discuss necesarry and sufficient conditions for the existence of spacelike stationary surfaces in $N_1^4(c)$ and $N_2^4(c)$, together with isometric deformations preserving normal curvature.

THEOREM 1 ([S2]). (i) Let M be a spacelike stationary surface in $N_1^4(c)$. We denote by K, K_ν and Δ the Gaussian curvature, the normal curvature and the Laplacian of M, respectively. Then

$$(1) \quad \Delta \log\{(c-K)^2 + K_\nu^2\} = 8K$$

at points where $(c-K)^2 + K_\nu^2 > 0$, and

$$(2) \quad \Delta \tan^{-1}\left(\frac{K_\nu}{c-K}\right) = -2K_\nu$$

at points where $K \neq c$.

(ii) Conversely, let M be a 2-dimensional simply connected Riemannian manifold with Gaussian curvature $K(\neq c)$ and Laplacian Δ. If K_ν is a function on M satisfying (1) and (2), then there exists an isometric stationary immersion of M into $N_1^4(c)$ with normal curvature K_ν.

THEOREM 2 ([S4]). Let $f: M \to N_1^4(c)$ be an isometric stationary immersion of a 2-dimensional simply connected Riemannian manifold M into $N_1^4(c)$ with nowhere vanishing normal curvature K_ν. Then there exists a 2π-periodic family of isometric stationary immersions $f_\theta : M \to N_1^4(c)$ with the same normal curvature K_ν. Moreover, if $\tilde{f} : M \to N_1^4(c)$ is another isometric stationary immersion with the same normal curvature K_ν, then there exists $\theta \in [0, \pi]$ such that $\tilde{f}$ and f_θ coincide up to congruence.

THEOREM 3 ([S3]). (i) Let M be a spacelike stationary surface in $N_2^4(c)$. We denote by K, K_ν and Δ the Gaussian curvature, the normal curvature and the Laplacian of M, respectively. Then

$$(3) \quad \Delta \log(K - c + K_\nu) = 2(2K + K_\nu)$$

and

$$(4) \quad \Delta \log(K - c - K_\nu) = 2(2K - K_\nu)$$

at non-isotropic points where $(K - c)^2 - K_\nu^2 > 0$.

(ii) Conversely, let M be a 2-dimensional simply connected Riemannian manifold with Gaussian curvature $K(> c)$ and Laplacian Δ. If K_ν is a function on M satisfying $(K - c)^2 - K_\nu^2 > 0$ and (3), (4), then there exists an isometric stationary immersion of M into $N_2^4(c)$ with normal curvature K_ν.

THEOREM 4 ([S3]). Let $f : M \to N_2^4(c)$ be a non-isotropic isometric stationary immersion of a 2-dimensional simply connected Riemannian manifold M into $N_2^4(c)$ with normal curvature K_ν. Then there exists a 2π-periodic family of isometric stationary immersions $f_\theta : M \to N_2^4(c)$ with the same normal curvature K_ν. Moreover, if $\tilde{f} : M \to N_2^4(c)$ is another isometric stationary immersion with the same normal curvature K_ν, then there exists $\theta \in [0, \pi]$ such that $\tilde{f}$ and f_θ coincide up to congruence.

THEOREM 5 ([S3]). (i) Let M be an isotropic spacelike stationary surface in $N_2^4(c)$ with Gaussian curvature K and Laplacian Δ. Then

$$(5) \quad \Delta \log(K - c) = 2(3K - c)$$

at points where $K > c$.

(ii) Conversely, let M be a 2-dimensional simply connected Riemannian manifold with Gaussian curvature $K(> c)$ and Laplacian Δ. If M satisfies (5), then there exists an isotropic isometric stationary immersion f of M into $N_2^4(c)$. Moreover, if $\tilde{f} : M \to N_2^4(c)$ is another isotropic isometric stationary immersion, then $\tilde{f}$ and f coincide up to congruence.

REMARK. For these theorems, see [GT] for the case of minimal surfaces in $N^4(c)$.

We discuss spacelike stationary surfaces in $N_2^4(c)$ with constant Gaussian curvature, or constant normal curvature. We also give a rigidity type theorem.

THEROEM 6 ([S3]). Let M be a spacelike stationary surface with constant Gaussian curvature K in $N_2^4(c)$. Then either (i) $K = c$ and M is totally geodesic, (ii) $c < 0$, $K = c/3$ and M is isotropic, or (iii) $c < 0$, $K = 0$ and M is congruent to a certain surface in a totally geodesic $N_1^3(c)$.

REMARK. Theorem 6 should be compared with [K] for minimal surfaces in $N^4(c)$.

THEROEM 7([S3]). Let M be a spacelike stationary surface with constant normal curvature K_ν in $N_2^4(c)$. Then either (i) M lies in a totally geodesic $N_1^3(c)$, or (ii) $c < 0$ and M has constant Gaussian curvature $c/3$.

THEOREM 8([S3]). Let M be a spacelike stationary surface in $N_2^4(c)$. If M is locally isometric to a spacelike stationary surface in $N_1^3(c)$, then M lies in a totally geodesic $N_1^3(c)$.

REMARK. For Theorem 8, see [S1] for the case of minimal surfaces in $N^4(c)$.

We give two classes of 2-dimensional Riemannian manifolds which can be realized as spacelike stationary surfaces in $N_p^n(c)$.

Let M be a 2-dimensional Riemannian manifold with Gaussian curvature K and Laplacian Δ. For each real number c, set

$$F_1^c = 2(K - c), \quad F_{p+1}^c = F_p^c + 2(p+1)K - \sum_{q=1}^{p} \Delta \log(F_q^c) \quad \text{if } F_p^c > 0.$$

THEOREM 9([S5]). Let M be a 2-dimensional simply connected Riemannian manifold. Suppose that $F_p^c > 0$ for $p < m$, and $F_m^c = 0$ identically. Then there exists an isometric stationary immersion of M into $N_{2[m/2]}^{2m}(c)$, where [ ] denotes the Gauss symbol.

THEOREM 10([S5]). Let M be a 2-dimensional simply connected Riemannian manifold with metric ds^2. Suppose that $F_p^c > 0$ for $p \le m$, and the metric $d\hat{s}^2 = \left(\prod_{p=1}^{m} F_p^c\right)^{1/(m+1)} ds^2$ is flat. Then there exists a 2π-periodic family of isometric stationary immersions of M into $N_m^{2m+1}(c)$.

REMARK. The conditions of Theorems 9 and 10 may be seen as generalized Ricci conditions (cf. [L1], [J]). There are many 2-dimensional Riemannian manifolds which satisfy the conditions.

COROLLARY ([S5]). For every positive integer m, there exists an isometric stationary immersion of the hyperbolic plane of constant curvature $-2/m(m+1)$ into $N^{2m}_{2[m/2]}(-1)$.

REMARK. (i) For every positive integer m, there exists an isometric minimal immersion of the 2-sphere of constant curvature $2/m(m+1)$ into the $2m$-dimensional unit sphere (cf. [C]).

(ii) The author does not know the explicit representations of the surfaces in the Corollary.

(iii) There exist many explicit flat spacelike stationary surfaces in pseudo-hyperbolic spaces (cf. [S5]).

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